Tutorial Session
Lecture 3: Key Theories of Structural Transformation

STEG Lecture Series on
Key Concepts in Macro Development

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Environment

Kongsamut-Rebelo-Xie (2001) Economy

- Intertemporal utility over total consumption:

$$\sum_{t=0}^{\infty} \beta^t \log C_t \tag{1}$$

where $\beta \in (0, 1)$ is the discount factor.

- Intratemporal utility over agriculture, manufacturing, and services consumption:

$$C_t = \omega_a \log (c_{at} - \bar{c}_a) + \omega_m \log (c_{mt}) + \omega_s \log (c_{st} + \bar{c}_s) \tag{2}$$

where $\omega_i > 0$, $\omega_a + \omega_m + \omega_s = 1$, and $\bar{c}_a, \bar{c}_s > 0$.

- Endowments in each period:
  - one unit of time;
  - a positive initial stock of capital, $K_0 > 0$.

- Capital accumulation:

$$K_{t+1} = (1 - \delta)K_t + X_t \tag{3}$$

  where $\delta \in [0, 1]$ is the depreciation rate and $X_t \geq 0$ is investment.

- Cobb–Douglas production functions for each good:

$$c_{it} = k_{it}^{\theta}(A_{it}n_{it})^{1-\theta}, \quad i \in \{a, m, s\} \tag{4}$$

$$X_t = k_{xt}^{\theta}(A_{xt}n_{xt})^{1-\theta} \tag{5}$$

- Assume constant sectoral TFP growth:

$$\frac{A_{it+1}}{A_{it}} = 1 + \gamma_i, \quad i \in \{a, m, s\}, \quad \text{and} \quad \frac{A_{xt+1}}{A_{xt}} = 1 + \gamma_x$$

- Capital and labor can be used in all sectors.

- Feasibility:

$$K_t \geq k_{at} + k_{mt} + k_{st} + k_{xt} \tag{6}$$

$$1 \geq n_{at} + n_{mt} + n_{st} + n_{xt} \tag{7}$$

Homework Assignment

Solve the following problem

1. Define a sequence-of-markets equilibrium in this economy.
2. Define an aggregate balanced growth path (ABGP) in this economy.
3. Show that there is an ABGP.
4. Show that along the ABGP the employment and expenditure shares
   (a) are constant for investment,
   (b) decrease for agriculture,
   (c) are constant for manufacturing,
   (d) increase for services.

1. Define a sequence-of-markets equilibrium in this economy

Problem of the Representative Firm

$$\max_{\{k_{it},n_{it}\}_{t=0}^{\infty}} \; p_{it}k_{it}^{\theta}(A_{it}n_{it})^{1-\theta} - w_t n_{it} - r_t k_{it}$$

$$\text{s.t.} \;\; k_{it}, n_{it} \geq 0$$

$$\max_{\{k_{xt},n_{xt}\}_{t=0}^{\infty}} \; k_{xt}^{\theta}(A_{xt}n_{xt})^{1-\theta} - w_t n_{xt} - r_t k_{xt}$$

$$\text{s.t.} \;\; k_{xt}, n_{xt} \geq 0$$

First Order Conditions (F.O.C.)

$$[k_{it}] : p_{it}\theta k_{it}^{\theta-1}(A_{it}n_{it})^{1-\theta} - r_t = 0 \tag{8}$$

$$[n_{it}] : p_{it}(1-\theta)k_{it}^{\theta}A_{it}^{1-\theta}n_{it}^{-\theta} - w_t = 0 \tag{9}$$

$$[k_{xt}] : \theta k_{xt}^{\theta-1}(A_{xt}n_{xt})^{1-\theta} - r_t = 0 \tag{10}$$

$$[n_{xt}] : (1-\theta)k_{xt}^{\theta}A_{xt}^{1-\theta}n_{xt}^{-\theta} - w_t = 0 \tag{11}$$

- From (8) and (10):

$$p_{it}\theta k_{it}^{\theta-1}(A_{it}n_{it})^{1-\theta} = \theta k_{xt}^{\theta-1}(A_{xt}n_{xt})^{1-\theta} \tag{12}$$

- From (9) and (11):

$$p_{it}(1-\theta)k_{it}^{\theta}A_{it}^{1-\theta}n_{it}^{-\theta} = (1-\theta)k_{xt}^{\theta}A_{xt}^{1-\theta}n_{xt}^{-\theta} \tag{13}$$

Equalization of capital-to-labor ratios

- Dividing (12) by (13):

$$\frac{k_{xt}}{n_{xt}} = \frac{k_{it}}{n_{it}} \tag{14}$$

- Multiplying and dividing each term on the left-hand-side of $\sum_i k_{it} + k_{xt} = K_t$ by its employment level:

$$\frac{k_{xt}}{n_{xt}} = \frac{k_{it}}{n_{it}} = K_t \tag{15}$$

Prices are pinned down by labor-augmenting technological progress

- Diving (11) by (9):

$$p_{it} = \underbrace{\left(\frac{k_{xt}}{n_{xt}}\frac{n_{it}}{k_{it}}\right)^{\theta}}_{=1 \text{ from (14)}} \left(\frac{A_{xt}}{A_{it}}\right)^{1-\theta} \Rightarrow$$

$$p_{it} = \left(\frac{A_{xt}}{A_{it}}\right)^{1-\theta} \tag{16}$$

Aggregation on the production side

$$Y_t = X_t + \sum_i p_{it} c_{it} \tag{17}$$

- Substituting p_{it} from (16) and using (15):

$$p_{it} c_{it} = p_{it} k_{it}^{\theta} (A_{it} n_{it})^{1-\theta} = K_t^{\theta} A_{xt}^{1-\theta} n_{it} \tag{18}$$

- Plugging this expression in (17) and because $\sum_i n_{it} + n_{xt} = 1$:

$$Y_t = K_t^{\theta} A_{xt}^{1-\theta} n_{xt} + \sum_i K_t^{\theta} A_{xt}^{1-\theta} n_{it} =^{\theta} A_{xt}^{1-\theta} K_t \tag{19}$$

Sectoral expenditures and employment

$$\frac{p_{it} c_{it}}{Y_t} = \frac{K_t^{\theta} A_{xt}^{1-\theta} n_{it}}{K_t^{\theta} A_{xt}^{1-\theta}} = \frac{n_{it}}{1} = \frac{n_{it}}{N_t} \tag{20}$$

Household Problem

$$\max_{\{c_{at},c_{mt},c_{st},K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log \left[\omega_a^{\frac{1}{\varepsilon}} (c_{at} - \bar{c}_a)^{\frac{\varepsilon-1}{\varepsilon}} + \omega_m^{\frac{1}{\varepsilon}} (c_{mt})^{\frac{\varepsilon-1}{\varepsilon}} + \omega_s^{\frac{1}{\varepsilon}} (c_{st} + \bar{c}_s)^{\frac{\varepsilon-1}{\varepsilon}} \right]^{\frac{\varepsilon}{\varepsilon-1}}$$

$$\text{s.t.} \quad p_{at}c_{at} + p_{mt}c_{mt} + p_{st}c_{st} + K_{t+1} = (1 - \delta + r_t)K_t + w_t$$

- I will solve the general problem for $\varepsilon \in [0, \infty]$.
- For now, assume the problem is well-defined and the solution is interior.
  - i.e. total consumption is large enough relative to $\bar{c}_a$ and $\bar{c}_s$.
  - See pp. 888–889 of the Handbook chapter for a necessary condition.

F.O.C. Consumption

$$[c_{at}] : \frac{1}{C_t}\omega_a(c_{at} - \bar{c}_a)^{-\frac{1}{\varepsilon}} C_t^{\frac{1}{\varepsilon}} = \lambda_t p_{at} \tag{21}$$

$$[c_{mt}] : \frac{1}{C_t}\omega_m(c_{mt})^{-\frac{1}{\varepsilon}} C_t^{\frac{1}{\varepsilon}} = \lambda_t p_{mt} \tag{22}$$

$$[c_{st}] : \frac{1}{C_t}\omega_s(c_{st} + \bar{c}_s)^{-\frac{1}{\varepsilon}} C_t^{\frac{1}{\varepsilon}} = \lambda_t p_{st} \tag{23}$$

- λ_t = current-value Lagrange multiplier on the budget constraint in t.
- Raising (21)–(23) to the power $(1 - \varepsilon)$, adding them and using the definition of C_t:

$$\frac{1}{C_t} = \lambda_t \left[\omega_a(p_{at})^{1-\varepsilon} + \omega_m(p_{mt})^{1-\varepsilon}\omega_s(p_{st})^{1-\varepsilon}\right]^{\frac{1}{1-\epsilon}}$$

- Because λ_t is the marginal value of an additional unit of expenditure in t:

$$P_t = \left[\omega_a (p_{at})^{1-\varepsilon} + \omega_m (p_{mt})^{1-\varepsilon} \omega_s (p_{st})^{1-\varepsilon}\right]^{\frac{1}{1-\epsilon}} \tag{24}$$

- Adding (21)–(23) and using this definition of P_t:

$$p_{at} c_{at} + p_{mt} c_{mt} + p_{st} c_{st} = P_t C_t + p_{at} \bar{c}_a - p_{st} \bar{c}_s \tag{25}$$

- We can split this problem into two sub-problems:

1. intertemporal: how to allocate total income between consumption and savings,
2. static: how to allocate consumption between the three consumption goods.

(i) Intertemporal Household Problem

$$\max_{\{C_t,K_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log C_t$$

$$\text{s.t.} \quad P_t C_t + K_{t+1} = (1 - \delta + r_t)K_t + w_t - p_{at}\bar{c}_a + p_{st}\bar{c}_s$$

F.O.C.

$$[C_t] : \frac{\beta^t}{C_t} = \mu_t P_t \tag{26}$$

$$[C_{t+1}] : \frac{\beta^{t+1}}{P_{t+1}C_{t+1}} = \mu_{t+1} \tag{27}$$

$$[K_{t+1}] : \mu_{t+1}(1 - \delta + r_t) = \mu_t \tag{28}$$

- where μ_t = Lagrange multiplier.

- Substituting (26) and (27) in (28):

$$\frac{P_{t+1}C_{t+1}}{P_t C_t} = \beta(1 - \delta + r_{t+1}) \tag{29}$$

- Transversality condition:

$$\lim_{T\to\infty} \beta^T \frac{1}{C_T} K_{t+1} = 0 \tag{30}$$

(ii) Static Household Problem (ε =1)

$$\max_{\{c_{at},c_{mt},c_{st}\}} \omega_a \log(c_{at} - \bar{c}_a) + \omega_a \log(c_{mt}) + \omega_s \log(c_{st} + \bar{c}_s)$$
$$\text{s.t.} \quad p_{at}c_{at} + p_{mt}c_{mt} + p_{st}c_{st} = P_t C_t + p_{at}\bar{c}_a - p_{st}\bar{c}_s$$

F.O.C.

$$[c_{at}] : \frac{\omega_a}{c_{at} - \bar{c}_a} = \gamma_t p_{at} \tag{31}$$

$$[c_{mt}] : \frac{\omega_m}{c_{mt}} = \gamma_t p_{mt} \tag{32}$$

$$[c_{st}] : \frac{\omega_s}{c_{st} + \bar{c}_s} = \gamma_t p_{st} \tag{33}$$

- Dividing (31) by (32):

$$\frac{p_{at}c_{at}}{p_{mt}c_{mt}} = \frac{\omega_a}{\omega_m} + \frac{p_{at}\bar{c}_a}{p_{mt}c_{mt}} \tag{34}$$

- Dividing (33) by (32):

$$\frac{p_{st}c_{st}}{p_{mt}c_{mt}} = \frac{\omega_a}{\omega_m} - \frac{p_{st}\bar{c}_s}{p_{mt}c_{mt}} \tag{35}$$

- Using the definition of P_tC_t, dividing by $p_{mt}c_{mt}$ and using the two conditions from above:

$$\frac{P_tC_t}{p_{mt}c_{mt}} = \frac{1}{\omega_m} \tag{36}$$

2. Define an Aggregate Balanced Growth Path

An Aggregate Balanced Growth Path (ABGP) implies:

1. Constant real interest rate: $r_t = r \ \forall t$.
2. Constant growth of capital per capita: $\frac{k_{t+1}}{k_t} = 1 + g_k \ \forall t$.
3. Constant growth of GDP per capita: $\frac{y_{t+1}}{y_t} = 1 + g_y \ \forall t$.
4. Constant capital-to-GDP ratio: $\frac{K_t}{Y_t} = b \ \forall t$.
5. Constant capital share: $\frac{r_t K_t}{Y_t} = s \ \forall t$.

3. Show that there is an ABGP

- Assume $r_t = r$.

- Because $\frac{k_{xt}}{n_{xt}} = K_t$, with the F.O.C. of the firm at t and $t+1$ we prove condition 2 of the ABGP:

$$\left(\frac{K_{t+1}}{K_t}\right)^{\theta-1}\left(\frac{A_{xt+1}}{A_{xt}}\right)^{1-\theta} = 1 \Rightarrow \frac{K_{t+1}}{K_t} = \frac{A_{xt+1}}{A_{xt}} = 1+\gamma_x \tag{37}$$

- Dividing Y_{t+1} by Y_t and using (19) we prove condition 3:

$$\frac{Y_{t+1}}{Y_t} = \left(\frac{A_{xt+1}}{A_{xt}}\right)^{1-\theta}\left(\frac{K_{xt+1}}{K_{xt}}\right)^{\theta} = (1+\gamma_x)^{1-\theta} + (1+\gamma_x)^{\theta} = 1+\gamma_x \tag{38}$$

- From the previous results, Y_t and K_t grow at the same rate $\Rightarrow$ we prove condition 4:

- Condition 5 follows from the Cobb-Douglas production technology:

$$\frac{rK_t}{Y_t} = \theta \tag{39}$$

Is $\mathbf{r_t = r\ \forall t}$?

- Dividing the law of motion of capital by K_t we show that X_t also grows at rate γ_x:

$$\frac{K_{t+1}}{K_t} = (1-\delta) + \frac{X_t}{K_t} \Rightarrow \frac{X_t}{K_t} = \gamma_x + \delta \tag{40}$$

- Because both Y_t and X_t grow at rate γ_x, then $P_t C_t$ grows at the same rate.

- Given this last condition and using the Euler equation (29) we show that r_t is constant:

$$r_{t+1} = r = \frac{1+\gamma_x}{\beta} - 1 + \delta \tag{41}$$

- Given a value of A_{x0} and the above condition, a unique value of K_0 exists along the ABGP:

$$\bar{K}_0 = \left[\frac{\beta\theta}{1+\gamma_x - \beta(1-\delta)}\right]^{\frac{1}{1-\theta}} A_{x0} \tag{42}$$

4. Employment and Expenditures Shares along the ABGP

- From the solution to the static household problem:

$$c_{at} = \frac{\omega_a P_t C_t}{p_{at}} + \bar{c}_a \tag{43}$$

$$c_{mt} = \frac{\omega_m P_t C_t}{p_{mt}} \tag{44}$$

$$c_{st} = \frac{\omega_s P_t C_t}{p_{st}} - \bar{c}_s \tag{45}$$

- Because $\frac{A_{it+1}}{A_{it}} = 1 + \gamma_i$, $\forall i \in \{a, m, s\}$: $\frac{p_{it}}{P_t} = \frac{p_{i0}}{P_0}$.

- Expressions (43)–(45) together with constant relative prices and the fact that $P_t C_t$ grows at rate $\gamma_x > 0$ imply:

$$\frac{c_{at+1}}{c_{at}} < \frac{c_{mt+1}}{c_{mt}} < \frac{c_{st+1}}{c_{st}} \tag{46}$$

- Thus, it follows that:

$$\frac{p_{at}c_{at}}{P_tC_t}\downarrow, \quad \frac{p_{mt}c_{mt}}{P_tC_t} =, \quad \frac{p_{st}c_{st}}{P_tC_t}\uparrow \tag{47}$$

- Also, we have that:

$$\frac{p_{it}c_{it}}{Y_t} = \frac{\left(\frac{A_{xt}}{A_{it}}\right)^{1-\theta} k_{it}^{\theta} A_{it}^{1-\theta} n_{it}^{1-\theta}}{K_t^{\theta} A_{xt}^{1-\theta}} = n_{it} \tag{48}$$

- Hence, because Y_t grows at rate γ_x:

$$n_{at}\downarrow, \quad n_{mt} =, \quad n_{st}\uparrow \tag{49}$$

- X_t and Y_t grow at rate $\gamma_x \Rightarrow$ employment and expenditures shares for investment are constant.

- See Proposition 2 of the Handbook Chapter (p. 890) for a condition on $\bar{c}_s$ that ensures the ABGP is well-defined.