Productivity and Wealth Dynamics under Financial Frictions: An Empirical Investigation of the Self-financing Channel

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The views expressed in this paper are our own and do not necessarily coincide with those of the Central Bank of Chile
Motivation

- Financial frictions prevent poor firms to invest at optimal scale
- If distributions of productivity and wealth are not aligned
  - Financial frictions → misallocation, low aggregate investment, TFP and income.
- Joint distribution of productivity and wealth drives quantitative results
- **Self-financing channel:** productive firms accumulate wealth and build collateral
  - Mitigate the effects of financial frictions on TFP (Moll 2014, Midrigan & Xu 2014)
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- Strength of self-financing depends on the **productivity process** and is reflected on how **policy functions** respond to productivity shocks
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- Strength of self-financing depends on the productivity process and is reflected on how policy functions respond to productivity shocks

- Two observations on previous literature
  1. Lack of micro evidence on policy functions and their response to productivity
  2. Prevalent methods to identify productivity fail under financial constraints
This Paper

- Characterizes and estimates the investment and wealth accumulation policy functions under financial fictions using firm-level data.
  - Examine the transmission of productivity shocks to investment and wealth accumulation and explore the self-financing channel using micro data.

- Uses rich micro data on firms’ balance sheet to
  1. Consistently estimate firm productivity process
     - How persistent and volatile are productivity shocks?
  2. Document the transmission of productivity shocks to firm decisions
     - How do these responses vary along the wealth distribution?
  3. Empirically explores the strength of the self-financing channel
     - How fast the MPKs of two firms with different levels of wealth converge?
New empirical approach

- Recover productivity from firm production function and estimate nonlinear policies that depends on key state variables
  - Consistent with heterogeneous-firm models with collateral constraints
    - (Moll 2014; Midrigan Xu 2014; Buera Kaboski Shin 2015, etc)
  - Not require specifying functional forms for preference and productivity
New empirical approach

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→ Challenge: productivity is not observable
  - Literature recovers productivity by estimating production function
    - Proxy Variable: Olley & Pakes 96; Levinsohn & Petrin 2003; etc
  - OP delivers biased estimates under financial frictions
  - Not well-suited to estimate policy functions

- Combine the insights of the self-financing channel with developments in nonlinear panel data to show non-parametric identification of the empirical model
  - (Hu & Schennach 2008, Arellano, Blundell & Bonhomme 2017)
Literature Review

Financial Frictions, Self-financing and Firms’ Dynamics:

- Banerjee Moll (2010); Buera Shin (2011); Buera Shin (2013); Buera Kaboski Shin (2013); Moll (2014); Midrigan Xu (2014); Buera Kaboski Shin (2015); Buera Kaboski Townsend (2021); Kaboski (2021) Drechsel (2022); di Giovanni Garcia-Santana Jeenas, Moral-Benito Pijoan-Mas (2022); Cavalcanti, Kaboski, Martin, Santos (2022)

- Banerjee Duflo (2005); Hall Jones (1999); Caggese (2007); Caggese Cunat (2008); Restuccia Rogerson (2008); Alburquerque Hoppenhayn (2004); Hsieh and Klenow (2009)

Production Function Estimation and Nonparametric Identification:

- Olley-Pakes (1996); Levinsohn Petrin (2003); Ackerberg, Caves, Frazer (2015); Gandhi, Navarro, Rivers (2020); Shenoy (2020)

- Hu Schennach (2008); Hu Shum (2012); Arellano, Blundell, Bonhomme (2017); Arellano Bonhomme (2017); Hu, Huang, Sasaki (2020); Cunha, Heckman, Schennach (2010)
Outline

✓ Introduction

► A simple model with Financial Frictions and Olley Pakes

3. Empirical framework

4. Results

5. Conclusion
A simple model (Buera, Kaboski & Shin 2015)

- Heterogenous entrepreneurs with initial net worth/wealth $A_{it}$ and prod $Z_{it}$

$$\max_{A_{it+1},K_{it+1},L_{it},C_{it}} \sum_{t=1}^{\infty} \beta^t E[u(C_{it})]$$

s.t. $$C_{it} + K_{it+1} - (1 - \delta)K_{it} = Y_{it} - W_t L_{it} + B_{it+1} - (1 + r_t)B_{it},$$

$$Y_{it} = Z_{it} K_{it}^{\beta_k} L_{it}^{\beta_l}$$

$$Z_{it} = \rho Z_{it-1} + \eta_{it}$$

$$A_{it} = K_{it} - B_{it}$$

- Financial friction: $B_{it} \leq \kappa(K_{it}, Z_{it}) \Rightarrow K_{it} \leq \tilde{\kappa}(A_{it}, Z_{it})$
A simple model (Buera, Kaboski & Shin 2015)

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\[
\begin{align*}
Y_{it} & = Z_{it}K_{it}^{\beta_k}L_{it}^{\beta_l} \\
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\end{align*}
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- Financial friction: $B_{it} \leq \kappa(K_{it}, Z_{it}) \Rightarrow K_{it} \leq \tilde{\kappa}(A_{it}, Z_{it})$

- Investment FOC

$$C_k E(Z_{it+1}|Z_{it})^{\frac{1}{1-\beta_l}} K_{it+1}^{\frac{\beta_k}{1-\beta_l} - 1} = \beta(r + \delta) + \mu(A_{it}, Z_{it})$$

$$\Rightarrow I_{i,t} = h_t(Z_{it}, A_{it})$$
A simple model (Buera, Kaboski & Shin 2015)
- Heterogenous entrepreneurs with initial net worth/wealth $A_{it}$ and prod $Z_{it}$

$$
\max_{A_{it+1}, K_{it+1, L_{it}, C_{it}}} \sum_{t=1}^{\infty} \beta^t E[u(C_{it})]
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$$
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$$
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- Investment FOC

$$
C_k E(Z_{it+1}|Z_{it}) \frac{1}{1-\beta_l} K_{it+1}^{\beta_k} = \beta(r + \delta) + \mu(A_{it}, Z_{it})
$$

$\Rightarrow I_{i,t} = h_t(Z_{it}, A_{it})$

- Self-financing (Euler):

$$
u'(C_{t}) = \beta [(1 + r_{t+1}) + E_t [\kappa A \hat{\mu}(A_{t+1}, Z_{t+1})]] u'(C_{t+1})
$$

$\Rightarrow A_{t+1} = g_{t+1}(Z_{it}, A_{it})$
Olley-Pakes under financial frictions

- Goal: Estimate the (log) production function:

\[ y_{it} = \beta_l l_{it} + \beta_k k_{it} + z_{it} + \varepsilon_{it} \]

- Endogeneity problem: \( \text{cov}(k_{i,t}, z_{i,t}) \neq 0, \text{cov}(l_{i,t}, z_{i,t}) \neq 0 \)
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- Intuition: high investment firms are proxied as high productive
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  ▶ Intuition: high investment firms are proxied as high productive

  ... but high investment firms might capture high wealth instead of high \( z \)!

- With financial constraints: \( i_{i,t} = h(z_{it}, a_{it}) \)

  ▶ In our simple model:

\[ y_{it} = \beta_l l_{it} + \beta_k k_{it} + \frac{1}{\rho}(1 - \beta_k - \beta_l)i_{it} + (1 - \beta_l)\tilde{\mu}(a_{it}) + \epsilon_{it} \]

  ▶ \( \text{Cov}(k_{it}\tilde{\mu}(a_{it})) < 0 \rightarrow \beta_k^{OP} < \beta_k \)

  ▶ High \( \tilde{\mu}(a_t) \rightarrow z^{OP} < z; \)

  ▶ \( \text{Cov}(l_{it}\tilde{\mu}(a_{it})) > 0 \rightarrow \beta_l^{OP} > \beta_l \)
Outline

✓ Introduction
✓ A simple model of financial frictions and Olley-Pakes
  ▶ Empirical Framework
4. Results
5. Conclusion
Empirical model under financial frictions

- Value added (log) production function for firms $i = 1 \cdots N,$ $t = 1 \cdots T$

$$y_{it} = \beta_l l_{it} + \beta_k k_{it} + z_{it} + \varepsilon_{it}$$

$$z_{it} = \varphi (z_{it-1}) + \eta_{it}$$

- The innovation $\eta_{it}$ is independent of $z_{it-1}$
- $k_{it}$ is dynamic but predetermined: $k_{it} = (1 - \delta) k_{t-1} + i_{it-1}$

- Non-linear policy functions

$$i_{it} = h_t (z_{it}, k_{it}, a_{it}, \mu_{it})$$

$$a_{it+1} = g_t (z_{it}, k_{it}, a_{it}, \nu_{it+1})$$

- $h(.)$ and $g(.)$ allow for rich interactions between $a_{it}$ (wealth) and $z_{it}$
- $\mu_{it}$ and $\nu_{it+1}$ are scalar i.i.d shocks independent of state variables at $t$
- The joint distribution of $z_{i1}, k_{i1}$ and $a_{i1}$ is left unrestricted
Objects of interest: marginal derivative effects

- Investment propensity in response to productivity shocks:
  \[ \Phi_t^h (a, k, z) = \mathbb{E}_\mu \left[ \frac{\partial h_t (z, k, a, \mu)}{\partial z} \right] \]

- Wealth accumulation propensity in response to productivity shocks:
  \[ \Phi_t^g (a, k, z) = \mathbb{E}_\nu \left[ \frac{\partial g_t (z, k, a, \nu_{t+1})}{\partial z} \right] \]

- Propensities reflect how firms react to productivity shocks

- Propensities are heterogeneous and vary with \( a_{it} \) and \( z_{it} \)
  - Evidence on collateral constraint: \( \Phi_t^h (a, k, z) \) increasing in \( a \)
  - Evidence on self-financing: \( \Phi_t^g (a, k, z) > 0 \) and decreasing in \( a \)
Identification Assumptions

- **Goal**: show that $\beta_k$, $\beta_l$, $\varphi(z_{it-1})$, $h_t$, $g_t$ are identified given that:
  - $y_{it}$, $k_{it}$, $l_{it}$, $i_{it}$, $a_{it}$ are observed for $i = 1 \cdots N$, $t = 1 \cdots T$
  - $z_{it}$, $\varepsilon_{it}$, $\nu_{it+1}$, $\mu_{it}$ are not observed
  - $z_{it}$ is correlated with $X_{it} = (l_{it}, a_{it}, k_{it})$.

- **Stochastic and Markovian assumptions**
  - $\varepsilon_{it}$, $\eta_{it}$, $\mu_{it}$ and $\nu_{it+1}$ are independent over time
  - $y_{it}$, $i_{it}$ and $a_{it+1}$ are independent given $\{k_{it}, a_{it}, z_{it}, l_{it}\}$
  - $a_{i}^{t+1}$ and $i_{i}^{t}$ are indep of $(a_{i}^{t-1}, k_{i}^{t-1}, z_{i}^{t-1})$ cond on $(a_{it}, k_{it}, z_{it})$
Nonparametric Identification

- Sequential identification scheme
  
  ▶ Identification of $\beta_k$ and $\beta_l$
  
  ▶ Given $\beta_k$ and $\beta_l$ we identify the productivity process and the policy functions
Intuition in linear model

- If \( f(a_{t+1} \mid z_t, X_t) \) and \( f(i_t \mid z_t, X_t) \) are normal distributed

\[
a_{it+1} = g_z z_{it} + g_a a_{it} + \nu_{it+1} \\
i_{it} = h_z z_{it} + h_a a_{it} + \mu_{it}
\]

- Self-financing \( \Rightarrow g_z \neq 0 \)
- Proxy variable: \( z_{it} = \pi_1 i_{it} + \pi_2 a_{it} + \pi_3 \mu_{it} \)

- Replacing the proxy variable in the prod function:

\[
y_{it} = \beta_l l_{it} + \beta_k k_{it} + \pi_1 i_{it} + \pi_2 a_{it} + \varepsilon_{it} + \pi_3 \mu_{it}
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\]

- Use \( a_{it+1} \) as an instrument for \( i_{it} \) in an IV estimation
  - Relevance: \( a_{it+1} \) is correlated with \( i_{it} \) through \( z_{it} \) if \( g_z \neq 0 \) (self-financing)
  - Exogeneity: \( E(i_{it}\mu_{it+1}) = 0 \) since \( E(\nu_{it+1}\mu_{it}) = 0 \)

- If a firm experiences a positive productivity shock it increases simultaneously investment and wealth accumulation
Non parametric identification (Hu Schennach 2008)

- Identify $f(y_{it} | z_{it}, k_{it}, l_{it}) \rightarrow E[y_{it} | z_{it} = 0, k_{it}, l_{it}] = \beta_{l} l_{it} + \beta_{k} k_{it}$

- Challenge: recover $f(y_{it} | z_{it}, k_{it}, l_{it})$ from $f(y_{it}, i_{it}, a_{it+1}, a_{it}, k_{it}, l_{it})$. 

Non parametric identification (Hu Schennach 2008)

- Identify \( f(y_{it} | z_{it}, k_{it}, l_{it}) \) → \( E[y_{it} | z_{it} = 0, k_{it}, l_{it}] = \beta_{l} l_{it} + \beta_{k} k_{it} \)

- Challenge: recover \( f(y_{it} | z_{it}, k_{it}, l_{it}) \) from \( f(y_{it}, i_{it}, a_{it+1}, a_{it}, k_{it}, l_{it}) \).

- Given the conditional independence assumption:

\[
f(y_{t}, i_{t} | a_{t+1}, X_t) = \int f(y_{t} | z_{t}, k_{t}, l_{t}) f(i_{t} | z_{t}, X_t) f(z_{t} | a_{t+1}, X_t) \, dz
\]
- Identify \( f(y_{it} | z_{it}, k_{it}, l_{it}) \) \( \rightarrow E[y_{it} | z_{it} = 0, k_{it}, l_{it}] = \beta_l l_{it} + \beta_k k_{it} \)

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\]

- Defining \( L_{y|z,k,l} \) and \( L_{a|z,X} \) as the integral operators associated with \( f(y_t | z_{it}, k_{it}, l_{it}) \) and \( f(a_{t+1} | z_{it}, X_t) \) and integrating over \( i_{it} \)

\[
L_{y;I|a,X} L_{y|a,X}^{-1} = L_{y|z,k,l} D I; z | X L_{y|z,k,l}^{-1}
\]

- Observed object \( L_{y;I|a,X} L_{y|a,X}^{-1} \) admits eigenvalue-eigenfunction decomp.
Non parametric identification (Hu Schennach 2008)

- Identify $f (y_{it} | z_{it}, k_{it}, l_{it}) \rightarrow E [y_{it} | z_{it} = 0, k_{it}, l_{it}] = \beta_l l_{it} + \beta_k k_{it}$

- Challenge: recover $f (y_{it} | z_{it}, k_{it}, l_{it})$ from $f (y_{it}, i_{it}, a_{it+1}, a_{it}, k_{it}, l_{it})$.

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$$f (y_t, i_t | a_{t+1}, X_t) = \int f (y_t | z_t, k_t, l_t) f (i_t | z_t, X_t) f (z_t | a_{t+1}, X_t) \, dz$$

- Defining $L_{y|z,k,l}$ and $L_{a|z,X}$ as the integral operators associated with $f (y_t | z_t, k_t, l_t)$ and $f (a_{t+1} | z_t, X_t)$ and integrating over $i_{it}$

$$L_{y;I|a,X} L_{y|a,X}^{-1} = L_{y|z,k,l} D_{I;z|X} L_{y|z,k,l}^{-1}$$

- Observed object $L_{y;I|a,X} L_{y|a,X}^{-1}$ admits eigenvalue-eigenfunction decomp.

  - $L_{y|a,X} = L_{y|z,k,l} L_{z|a,X}$ has to be invertible

  - Relevance: $L_{z|a,X}$ is invertible if $f (a_{t+1} | z_t, X_t)$ is complete
Identification Productivity Process

- Once we identify $\beta_l$ and $\beta_k$, we have the net income process:

\[
\tilde{y}_{it} = y_{it} - \beta_l l_{it} - \beta_k k_{it} = z_{it} + \epsilon_{it}
\]

- additive model with two independent latent variables $z_{it}$ and $\epsilon_{it}$
- Hidden Markov: $\{\tilde{y}_{it-2}, \tilde{y}_{it-1}, \tilde{y}_{it}\}$ are independent given $z_{it-1}$.
- the joint distribution of $(\epsilon_{i2}, \ldots, \epsilon_{iT-1})$ and $(z_{i2}, \ldots, z_{iT-1})$ are identified from the autocorrelation structure of $(\tilde{y}_{i1}, \ldots, \tilde{y}_{iT})$ for $T \geq 3$
Identification Productivity Process

- Once we identify $\beta_l$ and $\beta_k$, we have the net income process:

$$\tilde{y}_{it} = y_{it} - \beta_l l_{it} - \beta_k k_{it} = z_{it} + \varepsilon_{it}$$

- Additive model with two independent latent variables $z_{it}$ and $\varepsilon_{it}$
- Hidden Markov: $\{\tilde{y}_{it-2}, \tilde{y}_{it-1}, \tilde{y}_{it}\}$ are independent given $z_{it-1}$.
- The joint distribution of $(\varepsilon_{i2}, \varepsilon_{iT-1})$ and $(z_{i2}, \ldots, z_{iT-1})$ are identified from the autocorrelation structure of $(\tilde{y}_{i1}, \ldots, \tilde{y}_{iT})$ for $T \geq 3$

- Linear Process:

$$z_{it} = \rho_z z_{it-1} + \eta_{it}$$

$$\rightarrow \tilde{y}_{it} = \rho_z \tilde{y}_{it-1} + \eta_{it} + \varepsilon_{it} - \rho_z \varepsilon_{it-1}$$

- Use $\tilde{y}_{it-2}$ as an instrument for $\tilde{y}_{it-1}$ to identify $\rho_z$
- $\sigma_\eta^2$ and $\sigma_\varepsilon^2$ are identify from the following moment conditions:

$$E(\tilde{y}_{it}\tilde{y}_{it-1}) = \rho_z E(\tilde{y}_{it-1}\tilde{y}_{it-1}) - \rho_z \sigma_\varepsilon^2$$

$$E(\tilde{y}_{it}\tilde{y}_{it}) = \rho_z^2 E(\tilde{y}_{it-1}\tilde{y}_{it-1}) + \sigma_\eta^2 + (1 - \rho_z^2) \sigma_\varepsilon^2$$
Policy Functions: Linear case

- Policy rules depend on unobserved $z_{it}$:

$$a_{it+1} = g_z z_{it} + g_a a_{it} + g_k k_{it} + \nu_{it+1}$$

$$\rightarrow a_{it+1} = g_z \tilde{y}_{it} + g_a a_{it} + g_k k_{it} + \nu_{it+1} - g_z \varepsilon_{it}$$

- $E(\tilde{y}_{it}\varepsilon_{it}) \neq 0$: OLS regression of $a_{it+1}$ on $\tilde{y}_{it}$, $a_{it}$ and $k_{it}$ do not work

- Use $\tilde{y}_{it-1}$ as an instrument for $\tilde{y}_{it}$
  - Relevance: Markovian productivity ensures: $E(\tilde{y}_{it}\tilde{y}_{it-1}) \neq 0$
  - Exogeneity: $E(\tilde{y}_{it-1}\varepsilon_{it}) = 0$
Outline

✓ Introduction
✓ Brief example: Olley-Pakes under financial frictions
✓ Empirical framework
▶ Results
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Data

- Chilean tax annual administrative data from 2005-2016.
- Approx. 4800 firms (manufacturing), 13500 obs.
- Form 22 (income tax/ reflect dividend policies, balance sheet):
  - $y_{it}$: value added.
  - $k_{it}$: physical capital.
  - $a_{it}$: wealth/net worth.
- Form 29 (monthly report of expenditures/flows):
  - $i_{it}$: investment
- DJ 1887:
  - $l_{it}$: Number of workers
## Results: Production Function with Chilean data

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<tr>
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<th>OP</th>
<th>Proxy-IV</th>
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<tbody>
<tr>
<td>$\beta_k$</td>
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<td>$\beta_l$</td>
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</table>

### Table: Production Function Estimates from Microdata

Note: The table shows the Production function estimates from administrative data for Chile, using alternative methodologies: Olley and Pakes 1996 (OP), and two estimators that control for financial friction, Proxy-IV and SEM.
Investment Policy Rule: Nonlinear effects

- Investment propensity evaluated at different quantiles of $a$ and $z$
  - High heterogeneity in propensities (0.08-0.6)
  - Increasing in $a$ and $z$
  - Sensitivity to $a$ depends on $z$ (low $z$ increases in 200%, high $z$ in 30%)
Wealth Accumulation Policy Rule: Nonlinear
- Wealth accumulation propensity evaluated at different quantiles of $a$ and $z$
  - High heterogeneity in propensities (0.2 - 0.98)
  - Decreasing in $a$ and increasing $z$
  - Low $a$ - high $z$ firms benefit the most from an additional unit of wealth
How strong is self-financing?
- Simulate MPKs for two firms with the same $z$ but different $a$
- MPK of low $a$ firm is 3 times MPK of high $a$ (same as Banerjee and Moll 2010)
- Convergence takes more than 50 years (7 years in Banerjee and Moll 2010)

**Figure:** Convergence in MPKs
Conclusions

- Flexible non-linear framework to jointly model and estimate the firm wealth accumulation dynamics and the unobservable productivity process under collateral constraints.
- Reduce bias of prevalent strategies to estimate production functions and productivity.
- New results on policy functions: heterogenous responses of investment and wealth
  - Investment propensity is increasing in wealth and productivity
  - Wealth accumulation propensity is increasing in productivity and decreasing in wealth
- Self-financing is active in the data but its impact is limited
- Our estimates might inform structural models
  - Direct estimates of production parameters
  - New elasticities to indirectly estimate key deep parameters
Estimated propensities

- The figure exhibits how the investment propensity and the wealth accumulation propensity vary along the distribution of $\frac{A}{K}$ in the micro data.

- Each point represents the propensity of each particular firm evaluated at its actual value of $a$, $k$, and $z$.

Figure: Estimated propensities
Correlation between policy shocks

- For different values of $\beta_l$ and $\beta_k$ generate $\tilde{y}_{it}$:

$$\tilde{y}_{it} = y_{it} - \beta_l l_{it} - \beta_k k_{it} = z_{it} + \varepsilon_{it}$$

- Estimate using IV the following policies:

$$i_{it} = h_t(\tilde{y}_{it}, k_{it}, a_{it}) + \mu_{it}$$

$$a_{it+1} = g_t(\tilde{y}_{it}, k_{it}, a_{it}) + \nu_{it+1}$$

- Run OLS regression between $\mu_{it}$ and $\nu_{it+1}$
Empirical model under financial frictions

- At the beginning of $t = 1$ firm starts with fixed and financial assets $K_1$, $A_1^f$ and debt $D_1$ (decided in $t = 0$)

- Firm’s net worth at the beginning of $t = 1$ $A_1 = A_1^f + K_1 - D_1$

- Productivity $Z_1$ is realized:
  - Firm hires $L_1$ and produce with predetermined $K_1 \rightarrow Y_1$
  - There is time to built (is that it takes a full period for new capital to be ordered, delivered, and installed) so firm decides investment today to produce next period:
    - Use internal assets $A_1^f$ and/or ask for a loan: $K_2 \rightarrow I_1 = h(Z_1, A_1) \rightarrow \Delta D$
      $I_1$ is a production decision for $t = 2$ but depends on $A_1$ decided at $t = 0$

- At that point in time net worth is unchanged ($I_1 = \Delta D$)

- At the end of $t = 1$ firm decide to retain earnings or pay dividends
  $\Delta A^f = \tilde{Y}_1 - C_1 = \tilde{g}(E(u(A_2, Z_2))) = \tilde{g}(A_2, Z_1)$

- net worth in $t = 2$ (accumulated at the end of $t = 1$) is used for $I_2$ to produce in $t = 3$

- is the time to built assumption and the dividend payment policy at the end of the period that creates the timing that separates $I_1$ from $A_2$?
Investment Policy Rule: Linear effects

- Marginal effect of productivity and wealth on investment function evaluated at different quantiles of distribution of wealth

\[ i_{it} = h_1(t, k_{it}, a_{it}) + h_2(t, k_{it}, a_{it}) z_{it} + \nu_{it} \]

**Figure:** Marginal effect of productivity on investment
Materials Policy Rule: Linear effects

- Marginal effect of productivity on materials policy function evaluated at different quantiles of distribution of wealth

\[ m_{it} = h_{1t}(k_{it}, a_{it}) + h_{2t}(k_{it}, a_{it})z_{it} + \nu_{it} \]

**Figure**: Marginal effect of productivity on materials
Wealth Accumulation Policy Rule: Linear effects

- Marginal effect of productivity and wealth on wealth function evaluated at different quantiles of distribution of wealth

\[ a_{it+1} = g_{1t}(k_{it}, a_{it}) + g_{2t}(k_{it}, a_{it}) z_{it} + \mu_{it+1} \]

**Figure:** Marginal effect of productivity on investment
Results: Productivity Process

Figure: Estimated distribution of productivities

Notes: The figure shows the estimated distribution of firm-level productivities using administrative microdata for Chile, under alternative methodologies: OP, LP, the SEM algorithm using Normal shocks and the SEM algorithm using a quantile model (ABB).
Results: Productivity Process

Figure: Estimated distribution of productivities

Notes: The figure shows the estimated persistence of firm-level productivities using administrative microdata for Chile, using Normal shocks and the SEM algorithm using a quantile model (ABB).
## Results: Production Function with simulated data

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**Table: Production Function Estimates Using Simulated Data**

Note: Production function estimates from simulated data using alternative methodologies: Olley and Pakes 1996 (OP), and two estimators that control for financial friction, Proxy-IV and SEM.
Results: Production Function with Chilean data

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Table: Production Function Estimates from Microdata

Note: The table shows the Production function estimates from administrative data for Chile, using alternative methodologies: Olley and Pakes 1996 (OP), and two estimators that control for financial friction, Proxy-IV and SEM.
Results: Productivity Process

\[ z_{it} = \rho z_{it-1} + \eta_{it} \]

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<tr>
<td>( R^2 )</td>
<td>0.37</td>
<td>-</td>
<td>0.70</td>
</tr>
</tbody>
</table>
Investment Policy Rule: Nonlinear effects

- Marginal effect of productivity on nonlinear investment function evaluated at different quantiles of distribution of wealth

\[ i_{it} = h(z_{it}, k_{it}, a_{it}, \mu_{it}) \]
- Buera, Kaboski Townsend (JEL 2021) Over the same period of time, a macroeconomic literature has made advances in building and solving models incorporating rich micro-structure, that is, with well-defined agent problems, with heterogeneity, and with contracting and market frictions. However this line of work has tended to rely on strong structural assumptions, e.g., assumptions on functional forms and distributions of unobservables, and on somewhat stylized calibration strategies, and thus economists often view it as disconnected from micro empirical research.
A simple model with Financial Frictions

Entrepreneur, with initial wealth $A_t$, capital $K_t$ and productivity $Z_t$, maximize the discounted value of distributed profits, $D_t$:

$$V(A_t, K_t, Z_t) = \max_{A_{t+1}, K_{t+1}, L_t} D_t + \beta E[V(A_{t+1}, K_{t+1}, Z_{t+1})|Z_t]$$

subject to

$$D_t = Z_t K_t^{\beta_k} L_t^{\beta_l} - (r^b_t + \delta) K_t - w_t L_t - \frac{g(A_{t+1})}{1 + r^a_t} + A_t$$

$$K_{t+1} \leq \lambda A_t$$

where $g(\cdot)$ is a cost function and $\lambda$ is the financial friction parameter.
A simple model with Financial Frictions

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$$st. \quad D_t = Z_t K_t^{\beta_k} L_t^{\beta_l} - (r_t^b + \delta) K_t - w_t L_t - \frac{g(A_{t+1})}{1 + r_t^a} + A_t$$

$$K_{t+1} \leq \lambda A_t$$

where $g(\cdot)$ is a cost function and $\lambda$ is the financial friction parameter.

- FOCs

$$\beta_l Z_t K_t^{\beta_k} L_t^{\beta_l-1} = w + \omega_t$$

$$C_k E(Z_{t+1} | Z_t)^{\frac{1}{1-\beta_l}} K_t^{\frac{\beta_k}{1-\beta_l}-1} = r + \delta + \mu(A_t)$$

$$g'(A_{t+1}) = \beta (1 + \lambda E[\mu(A_{t+1})])$$

where $\mu(A_t)$ is the multiplier of the financial constraint.
Estimating Capital Elasticity

- From the FOC for investment we obtain an implicit function for $z_t$ (after taking logs and assuming a Markov process for $z_t$)

$$z_t = c_z + \frac{1}{\rho}((1 - \beta_k - \beta_l)k_{t+1} + (1 - \beta_l)\mu(a_t))$$

- Following OP we replace this into the production function

$$y_t = \beta_l l_t + \beta_k k_t + z_t + \varepsilon_t$$

$$= \beta_l l_t + \beta_k k_t + \frac{1}{\rho}(1 - \beta_k - \beta_l)k_{t+1} + (1 - \beta_l)\mu(a_t) + \varepsilon_t$$

- OP: investment ($k_{t+1}$ here) controls for unobservable productivity
- But with collateral constraints $E(k_t\mu(a_t)) < 0 \rightarrow \beta_{k}^{OP} < \beta_k$
Estimating Labor Elasticity

- We can express the FOC for labor as
  \[ l_t = c_n + \frac{1}{1 - \beta_l} (\beta_k k_t + \tilde{\omega}_t + z_t) \]
  \[ = \tilde{c}_n + \frac{1}{1 - \beta_l} (\beta_k k_t + \tilde{\omega}_t) + \frac{(1 - \beta_k - \beta_l)}{\rho(1 - \beta_l)} k_{t+1} + \frac{1}{\rho} \tilde{\mu}(a_t) \]

- Recall the equation for estimating the production function
  \[ y_t = \beta_l l_t + \beta_k k_t + z_t + \varepsilon_t \]
  \[ = \beta_l l_t + \beta_k k_t + \frac{1}{\rho} (1 - \beta_k - \beta_l) k_{t+1} + (1 - \beta_l) \tilde{\mu}(a_t) + \varepsilon_t \]

- With collateral constraints \( E(l_t \tilde{\mu}(a_t)) > 0 \rightarrow \beta_l^{OP} > \beta_l \)
Estimation: Policies without shocks

• OPA: Augmented OP

\[ i_{it} = h(z_{it}, k_{it}, a_{it}) \Rightarrow z_{it} = h^{-1}(i_{it}, k_{it}, a_{it}) \]

\[ \Rightarrow y_{it} = \beta_l l_{it} + \beta_k k_{it} + h^{-1}(i_{it}, k_{it}, a_{it}) + \varepsilon_{it}, \]

▷ OLS of \( y_{it} \) on \( l_{it} \) and a polynomial on \( k_{it}, i_{it} \) and \( a_{it} \)
Estimation: Policies without shocks

- OPA: Augmented OP
  \[ i_{it} = h(z_{it}, k_{it}, a_{it}) \Rightarrow z_{it} = h^{-1}(i_{it}, k_{it}, a_{it}) \]

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  \[ \text{OLS of } y_{it} \text{ on } l_{it} \text{ and a polynomial on } k_{it}, i_{it} \text{ and } a_{it} \]

- Proxy-Wealth
  \[ a_{it+1} = g(z_{it}, k_{it}, a_{it}) \Rightarrow z_{it} = g^{-1}(a_{it+1}, k_{it}, a_{it}) \]

  \[ \Rightarrow y_{it} = \beta_l l_{it} + \beta_k k_{it} + g^{-1}(a_{it+1}, k_{it}, a_{it}) + \varepsilon_{it}, \]
  \[ \text{OLS of } y_{it} \text{ on } l_{it} \text{ and a polynomial on } k_{it}, a_{it+1} \text{ and } a_{it} \]
Policies with shocks: Proxy-IV

- Consider the following quasi-linear wealth accumulation policy:

\[ a_{it+1} = g_1(k_{it}, a_{it}) + g_2(k_{it}, a_{it}) z_{it} + w_{it+1} \]

\[ \Rightarrow \]

\[ z_{it} = \pi_1(k_{it}, a_{it}) + \pi_2(k_{it}, a_{it}) a_{it+1} + \omega_{it+1} \]
Policies with shocks: Proxy-IV

- Consider the following quasi-linear wealth accumulation policy:

\[ a_{it+1} = g_1 (k_{it}, a_{it}) + g_2 (k_{it}, a_{it}) z_{it} + w_{it+1} \]

\[ \Rightarrow \]

\[ z_{it} = \pi_1 (k_{it}, a_{it}) + \pi_2 (k_{it}, a_{it}) a_{it+1} + \omega_{it+1} \]

- Replacing the proxy variable in the production function:

\[ y_{it} = \beta_l l_{it} + \phi (k_{it}, a_{it}) + \pi_2 (k_{it}, a_{it}) a_{it+1} + \omega_{it+1} + \varepsilon_{it} \]

where \( \phi (k_{it}, a_{it}) = \beta_k k_{it} + \pi_1 (k_{it}, a_{it}) \)

- IV using \( \pi_2 (k_{it}, a_{it}) i_{it} \) as the instrument for \( \pi_2 (k_{it}, a_{it}) a_{it+1} \)
### Results: Production Function

<table>
<thead>
<tr>
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<th>OPA</th>
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- OP: inverting the investment equation
- OPA: inverting the investment equation but controlling for $a_{it}$
- Proxy-Wealth: inverting the wealth accumulation equation
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- SEM: non-linear approach that uses the full information of the model
### Results: Production Function with Chilean data

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Results: Productivity Process

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Estimator

- Unfeasible moment condition:

\[
E \left[ \sum_{t=1}^{T} \left( y_{it} - \beta_l l_{it} - \beta_k k_{it-1} - z_{it} \right) k_{it-1} \right] = 0
\]

- Applying the law of iterated exp, obtain the integrated moment conditions:

\[
E \left[ \int \left( \sum_{t=1}^{T} \left( y_{it} - \beta_l l_{it} - \beta_k k_{it-1} - z_{it} \right) k_{it-1} \right) f\left( z_{it} \mid z_{it-1}, y_{it}, k_{it-1}, l_{it} \right) dz \right] = 0
\]

where \( f\left( z_{it} \mid z_{it-1}, y_{it}, k_{it-1}, l_{it} \right) \) is the posterior distribution of \( z_{it} \) given data.

- Draw \( \{ z_{it}^{(1)} \ldots z_{it}^{(M)} \} \) \( M \) realizations of \( z_{it} \) from \( f\left( z_{it} \mid .\right) \)

\[
\sum_{i=1}^{N} \sum_{m=1}^{M} \sum_{t=1}^{T} \left( \left( y_{it} - \beta_l l_{it} - \beta_k k_{it-1} - z_{it}^{(m)} \right) k_{it-1} \right)
\]
First step: posterior distributions

- Let $X_i$ and $Z_i$ the history of observables and productivity for firm $i$.
- Posterior distribution of $z_{it}$

$$
 f (z_{i1} \ldots z_{iT}, X_i) = \prod_{t=1}^{T} f (y_{it} \mid k_{it}, l_{it}, z_{it}) \times \\
 \prod_{t=1}^{T} f (i_{it} \mid k_{it}, z_{it}, a_{it}) \times \\
 \prod_{t=2}^{T} f (a_{it} \mid z_{it-1}, k_{it-1}, i_{it-1}, a_{it-1}) f (a_{i1} \mid z_{i1}) \times \\
 \prod_{t=1}^{T} f (z_{it} \mid z_{it-1}) f (z_{i1})
$$
Second step: Integrated GMM

- After drawing $M$ realizations of $Z_{i}^{(m)}$ from the joint posterior distribution, compute the integrated-GMM estimator of the parameters:

\[
\sum_{i=1}^{N} \sum_{t=2}^{T} \sum_{m=1}^{M} \left( y_{it} - \beta_{l_{it}} - \beta_{k_{it}} - z_{it}^{(m)} \right)^{2}
\]

\[
\sum_{i=1}^{N} \sum_{t=2}^{T} \sum_{m=1}^{M} \left( i_{it} - \tilde{h} \left( z_{it}^{(m)}, k_{it}, a_{it} \right) \right)^{2}
\]

\[
\sum_{i=1}^{N} \sum_{t=2}^{T} \sum_{m=1}^{M} \left( a_{it+1} - \tilde{g} \left( z_{it}^{(m)}, a_{it}, k_{it}, i_{it} \right) \right)^{2}
\]

\[
\sum_{i=1}^{N} \sum_{t=2}^{T} \sum_{m=1}^{M} \left( z_{it}^{(m)} - \tilde{\varphi} \left( z_{it-1}^{(m)} \right) \right)^{2}
\]

where $\tilde{h}(.)$, $\tilde{g}(.)$, $\tilde{\varphi}(.)$, $\tilde{\pi}(.)$ are the polynomial approx. of $h(.)$, $g(.)$, $\varphi(.)$, $\pi(.)$.  

Estimation: No shocks in the policies

- OPA: Augmented OP (including $a_{it}$ in investment)

$$z_{it} = h^{-1}(i_{it}, k_{it}, a_{it})$$
Estimation: No shocks in the policies

- OPA: Augmented OP (including $a_{it}$ in investment)
  
  $$ z_{it} = h^{-1}(i_{it}, k_{it}, a_{it}) $$

  ▶ First Stage: Replace proxy variable in the production function
  
  $$ y_{it} = \beta_l l_{it} + \phi(i_{it}, k_{it}, a_{it}) + \varepsilon_{it}, $$

  where $\phi(i_{it}, k_{it}, a_{it}) = \beta_k k_{it} + h^{-1}(i_{it}, k_{it}, a_{it})$.

  ▶ OLS of $y_{it}$ on $l_{it}$ and a polynomial identifies $\beta_l$ and $\phi(i_{it}, k_{it}, a_{it})$
Estimation: No shocks in the policies

- OPA: Augmented OP (including $a_{it}$ in investment)

$$z_{it} = h^{-1}(i_{it}, k_{it}, a_{it})$$

- First Stage: Replace proxy variable in the production function

$$y_{it} = \beta l_{it} + \phi(i_{it}, k_{it}, a_{it}) + \epsilon_{it},$$

where $\phi(i_{it}, k_{it}, a_{it}) = \beta k_{it} + h^{-1}(i_{it}, k_{it}, a_{it})$.

  - OLS of $y_{it}$ on $l_{it}$ and a polynomial identifies $\beta_l$ and $\phi(i_{it}, k_{it}, a_{it})$

- Second Stage: Use markovian model of productivity

$$\phi_t(i_{it}, k_{it}, a_{it}) = \beta k_{it} + \rho_z \phi_t(i_{it-1}, k_{it-1}, a_{it-1}) - \rho_z \beta k_{it-1} + \eta_{it} + \epsilon_{it}$$

  - OLS identifies $\beta_k$
Estimation: No shocks in the policies

• OPA: Augmented OP (including $a_{it}$ in investment)

$$z_{it} = h^{-1}(i_{it}, k_{it}, a_{it})$$

▷ First Stage: Replace proxy variable in the production function

$$y_{it} = \beta_l l_{it} + \phi(i_{it}, k_{it}, a_{it}) + \varepsilon_{it},$$

where $\phi(i_{it}, k_{it}, a_{it}) = \beta_k k_{it} + h^{-1}(i_{it}, k_{it}, a_{it})$.

▷ OLS of $y_{it}$ on $l_{it}$ and a polynomial identifies $\beta_l$ and $\phi(i_{it}, k_{it}, a_{it})$

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$$\phi_t(i_{it}, k_{it}, a_{it}) = \beta_k k_{it} + \rho_z \phi_t(i_{it-1}, k_{it-1}, a_{it-1}) - \rho_z \beta_k k_{it-1} + \eta_{it} + \varepsilon_{it}$$

▷ OLS identifies $\beta_k$

• Proxy-Wealth: Invert the wealth accumulation policy function

$$z_{it} = g^{-1}(a_{it+1}, k_{it}, a_{it})$$
Estimation: Policy functions with shocks

- Proxy-IV: Consider the following quasi-linear wealth policy function

\[ a_{it+1} = g_1(k_{it}, a_{it}) + g_2(k_{it}, a_{it}) z_{it} + w_{it+1} \]

\[ \Rightarrow \]

\[ z_{it} = \pi_1(k_{it}, a_{it}) + \pi_2(k_{it}, a_{it}) a_{it+1} + \omega_{it+1} \]
Estimation: Policy functions with shocks

- Proxy-IV: Consider the following quasi-linear wealth policy function

\[ a_{it+1} = g_1(k_{it}, a_{it}) + g_2(k_{it}, a_{it}) z_{it} + w_{it+1} \]

\[ \Rightarrow \]

\[ z_{it} = \pi_1(k_{it}, a_{it}) + \pi_2(k_{it}, a_{it}) a_{it+1} + \omega_{it+1} \]

- First Stage:

\[ y_{it} = \beta l_{it} + \phi(k_{it}, a_{it}) + \pi_2(k_{it}, a_{it}) a_{it+1} + \omega_{it+1} + \varepsilon_{it} \]

where \( \phi(k_{it}, a_{it}) = \beta_k k_{it} + \pi_1(k_{it}, a_{it}) \)

- IV using \( \pi_2(k_{it}, a_{it}) i_{it} \) as the instrument for \( \pi_2(k_{it}, a_{it}) a_{it+1} \)
Estimation: Policy functions with shocks

- Proxy-IV: Consider the following quasi-linear wealth policy function

\[ a_{it+1} = g_1(k_{it}, a_{it}) + g_2(k_{it}, a_{it}) z_{it} + w_{it+1} \]

\[ z_{it} = \pi_1(k_{it}, a_{it}) + \pi_2(k_{it}, a_{it}) a_{it+1} + \omega_{it+1} \]

- First Stage:

\[ y_{it} = \beta_l l_{it} + \phi(k_{it}, a_{it}) + \pi_2(k_{it}, a_{it}) a_{it+1} + \omega_{it+1} + \varepsilon_{it} \]

where \( \phi(k_{it}, a_{it}) = \beta_k k_{it} + \pi_1(k_{it}, a_{it}) \)

- IV using \( \pi_2(k_{it}, a_{it}) i_{it} \) as the instrument for \( \pi_2(k_{it}, a_{it}) a_{it+1} \)

- Second Stage:

\[ z_{it} = \rho_z \pi_1(k_{it-1}, a_{it-1}) + \rho_z \pi_2(k_{it-1}, a_{it-1}) a_{it} + \rho_z \omega_{it} + \eta_{it}, \]