Tutorial Session
Lecture 3: Key Theories of Structural Transformation

STEG Lecture Series on
Key Concepts in Macro Development

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Environment

Kongsamut-Rebelo-Xie (2001) Economy

- Intertemporal utility over total consumption:

\[ \sum_{t=0}^{\infty} \beta^t \log C_t \]  

where \( \beta \in (0, 1) \) is the discount factor.

- Intratemporal utility over agriculture, manufacturing, and services consumption:

\[ C_t = \omega_a \log (c_{at} - \bar{c}_a) + \omega_m \log (c_{mt}) + \omega_s \log (c_{st} + \bar{c}_s) \]  

where \( \omega_i > 0, \omega_a + \omega_m + \omega_s = 1 \), and \( \bar{c}_a, \bar{c}_s > 0 \).
• Endowments in each period:
  ○ one unit of time;
  ○ a positive initial stock of capital, $K_0 > 0$.

• Capital accumulation:

\[ K_{t+1} = (1 - \delta)K_t + X_t \]  

where $\delta \in [0, 1]$ is the depreciation rate and $X_t \geq 0$ is investment.
• Cobb–Douglas production functions for each good:

\[ c_{it} = k^\theta_{it} (A_{it} n_{it})^{1-\theta}, \quad i \in \{a, m, s\} \]  
\[ X_t = k^\theta_{xt} (A_{xt} n_{xt})^{1-\theta} \]  

\[ c_{it} = k^\theta_{it} (A_{it} n_{it})^{1-\theta}, \quad i \in \{a, m, s\} \]  
\[ X_t = k^\theta_{xt} (A_{xt} n_{xt})^{1-\theta} \]  

• Assume constant sectoral TFP growth:

\[ \frac{A_{it+1}}{A_{it}} = 1 + \gamma_i, \quad i \in \{a, m, s\}, \quad \text{and} \quad \frac{A_{xt+1}}{A_{xt}} = 1 + \gamma_x \]

• Capital and labor can be used in all sectors.

• Feasibility:

\[ K_t \geq k_{at} + k_{mt} + k_{st} + k_{xt} \]  
\[ 1 \geq n_{at} + n_{mt} + n_{st} + n_{xt} \]
Homework Assignment

Solve the following problem

1. Define a sequence-of-markets equilibrium in this economy.

2. Define an aggregate balanced growth path (ABGP) in this economy.

3. Show that there is an ABGP.

4. Show that along the ABGP the employment and expenditure shares
   (a) are constant for investment,
   (b) decrease for agriculture,
   (c) are constant for manufacturing,
   (d) increase for services.
1. Define a sequence-of-markets equilibrium in this economy
Problem of the Representative Firm

\[
\max_{\{k_{it}, n_{it}\}_{t=0}^{\infty}} \quad p_{it}k_{it}^\theta (A_{it}n_{it})^{1-\theta} - w_t n_{it} - r_t k_{it}
\]
\[\text{s.t.} \quad k_{it}, n_{it} \geq 0\]

\[
\max_{\{k_{xt}, n_{xt}\}_{t=0}^{\infty}} \quad k_{xt}^\theta (A_{xt}n_{xt})^{1-\theta} - w_t n_{xt} - r_t k_{xt}
\]
\[\text{s.t.} \quad k_{xt}, n_{xt} \geq 0\]
First Order Conditions (F.O.C.)

\[
[k_{it}] : p_{it} \theta k_{it}^{\theta-1} (A_{it} n_{it})^{1-\theta} - r_t = 0 \tag{8}
\]

\[
[n_{it}] : p_{it} (1 - \theta) k_{it}^{\theta} A_{it}^{1-\theta} n_{it}^{-\theta} - w_t = 0 \tag{9}
\]

\[
[k_{xt}] : \theta k_{xt}^{\theta-1} (A_{xt} n_{xt})^{1-\theta} - r_t = 0 \tag{10}
\]

\[
[n_{xt}] : (1 - \theta) k_{xt}^{\theta} A_{xt}^{1-\theta} n_{xt}^{-\theta} - w_t = 0 \tag{11}
\]

• From (8) and (10):

\[
p_{it} \theta k_{it}^{\theta-1} (A_{it} n_{it})^{1-\theta} = \theta k_{xt}^{\theta-1} (A_{xt} n_{xt})^{1-\theta} \tag{12}
\]

• From (9) and (11):

\[
p_{it} (1 - \theta) k_{it}^{\theta} A_{it}^{1-\theta} n_{it}^{-\theta} = (1 - \theta) k_{xt}^{\theta} A_{xt}^{1-\theta} n_{xt}^{-\theta} \tag{13}
\]
Equalization of capital-to-labor ratios

- Dividing (12) by (13):
  \[ \frac{k_{xt}}{n_{xt}} = \frac{k_{it}}{n_{it}} \]
  \[(14)\]

- Multiplying and dividing each term on the left-hand-side of \( \sum_i k_{it} + k_{xt} = K_t \) by its employment level:
  \[ \frac{k_{xt}}{n_{xt}} = \frac{k_{it}}{n_{it}} = K_t \]
  \[(15)\]

Prices are pinned down by labor-augmenting technological progress

- Diving (11) by (9):
  \[ p_{it} = \left( \frac{k_{xt} n_{it}}{n_{xt} k_{it}} \right)^\theta \left( \frac{A_{xt}}{A_{it}} \right)^{1-\theta} \]
  \[=1 \text{ from } (14) \]
  \[ p_{it} = \left( \frac{A_{xt}}{A_{it}} \right)^{1-\theta} \]
  \[(16)\]
Aggregation on the production side

\[ Y_t = X_t + \sum_i p_{it}c_{it} \quad (17) \]

- Substituting \( p_{it} \) from (16) and using (15):

\[ p_{it}c_{it} = p_{it}k_{it}^\theta(A_{it}n_{it})^{1-\theta} = K_t^\theta A_x^{1-\theta} n_{it} \quad (18) \]

- Plugging this expression in (17) and because \( \sum_i n_{it} + n_{xt} = 1 \):

\[ Y_t = K_t^\theta A_x^{1-\theta} n_{xt} + \sum_i K_t^\theta A_x^{1-\theta} n_{it} = \theta A_x^{1-\theta} K_t \quad (19) \]

Sectoral expenditures and employment

\[ \frac{p_{it}c_{it}}{Y_t} = \frac{K_t^\theta A_x^{1-\theta} n_{it}}{K_t^\theta A_x^{1-\theta}} = \frac{n_{it}}{1} = \frac{n_{it}}{N_t} \quad (20) \]
Household Problem

\[
\max_{\{c_{at}, c_{mt}, c_{st}, K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log \left[ \omega_a^\varepsilon (c_{at} - \bar{c}_a)^\frac{\varepsilon-1}{\varepsilon} + \omega_m^\varepsilon (c_{mt})^\frac{\varepsilon-1}{\varepsilon} + \omega_s^\varepsilon (c_{st} + \bar{c}_s)^\frac{\varepsilon-1}{\varepsilon} \right]^\frac{\varepsilon}{\varepsilon-1}
\]

s.t. \( p_{at}c_{at} + p_{mt}c_{mt} + p_{st}c_{st} + K_{t+1} = (1 - \delta + r_t)K_t + w_t \)

- I will solve the general problem for \( \varepsilon \in [0, \infty] \).

- For now, assume the problem is well-defined and the solution is interior.
  - i.e. total consumption is large enough relative to \( \bar{c}_a \) and \( \bar{c}_s \).
F.O.C. Consumption

\[
[c_{at}] : \frac{1}{C_t} \omega_a (c_{at} - \bar{a})^{\frac{1}{\epsilon}} C_t^{\frac{1}{\epsilon}} = \lambda_t p_{at}
\]  (21)

\[
[c_{mt}] : \frac{1}{C_t} \omega_m (c_{mt})^{\frac{1}{\epsilon}} C_t^{\frac{1}{\epsilon}} = \lambda_t p_{mt}
\]  (22)

\[
[c_{st}] : \frac{1}{C_t} \omega_s (c_{st} + \bar{s})^{\frac{1}{\epsilon}} C_t^{\frac{1}{\epsilon}} = \lambda_t p_{st}
\]  (23)

- \( \lambda_t \) = current-value Lagrange multiplier on the budget constraint in \( t \).

- Raising (21)–(23) to the power \((1 - \epsilon)\), adding them and using the definition of \( C_t \):

\[
\frac{1}{C_t} = \lambda_t \left[ \omega_a (p_{at})^{1-\epsilon} + \omega_m (p_{mt})^{1-\epsilon} \omega_s (p_{st})^{1-\epsilon} \right]^{\frac{1}{1-\epsilon}}
\]
• Because $\lambda_t$ is the marginal value of an additional unit of expenditure in $t$:

$$P_t = \left[ \omega_a(p_{at})^{1-\varepsilon} + \omega_m(p_{mt})^{1-\varepsilon} \omega_s(p_{st})^{1-\varepsilon} \right]^{1/1-\varepsilon}$$  \hspace{1cm} (24)

• Adding (21)–(23) and using this definition of $P_t$:

$$p_{at}c_{at} + p_{mt}c_{mt} + p_{st}c_{st} = P_tC_t + p_{at}\bar{c}_a - p_{st}\bar{c}_s$$  \hspace{1cm} (25)

• We can split this problem into two sub-problems:
  1. intertemporal: how to allocate total income between consumption and savings,
  2. static: how to allocate consumption between the three consumption goods.
(i) Intertemporal Household Problem

\[
\max_{\{C_t, K_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log C_t \\
\text{s.t. } P_t C_t + K_{t+1} = (1 - \delta + r_t) K_t + w_t - p_{at} \bar{c}_a + p_{st} \bar{c}_s
\]

F.O.C.

\[
[C_t] : \frac{\beta^t}{C_t} = \mu_t P_t \tag{26}
\]

\[
[C_{t+1}] : \frac{\beta^{t+1}}{P_{t+1} C_{t+1}} = \mu_{t+1} \tag{27}
\]

\[
[K_{t+1}] : \mu_{t+1}(1 - \delta + r_t) = \mu_t \tag{28}
\]

- where \( \mu_t = \) Lagrange multiplier.
• Substituting (26) and (27) in (28):

\[
\frac{P_{t+1}C_{t+1}}{P_tC_t} = \beta (1 - \delta + r_{t+1})
\]

(29)

• Transversality condition:

\[
\lim_{T \to \infty} \beta^T \frac{1}{C_T} K_{t+1} = 0
\]

(30)
(ii) Static Household Problem ($\varepsilon = 1$)

$$\max_{\{c_{at}, c_{mt}, c_{st}\}} \omega_a \log(c_{at} - \bar{c}_a) + \omega_a \log(c_{mt}) + \omega_s \log(c_{st} + \bar{c}_s)$$

s.t. $p_{at}c_{at} + p_{mt}c_{mt} + p_{st}c_{st} = P_tC_t + p_{at}\bar{c}_a - p_{st}\bar{c}_s$

F.O.C.

$$[c_{at}] : \frac{\omega_a}{c_{at} - \bar{c}_a} = \gamma_t p_{at} \quad (31)$$

$$[c_{mt}] : \frac{\omega_m}{c_{mt}} = \gamma_t p_{mt} \quad (32)$$

$$[c_{st}] : \frac{\omega_s}{c_{st} + \bar{c}_s} = \gamma_t p_{st} \quad (33)$$
• Dividing (31) by (32):

\[
\frac{p_{at}c_{at}}{p_{mt}c_{mt}} = \frac{\omega_a}{\omega_m} + \frac{p_{at}\bar{c}_a}{p_{mt}c_{mt}}
\]  

(34)

• Dividing (33) by (32):

\[
\frac{p_{st}c_{st}}{p_{mt}c_{mt}} = \frac{\omega_a}{\omega_m} - \frac{p_{st}\bar{c}_s}{p_{mt}c_{mt}}
\]  

(35)

• Using the definition of \( P_tC_t \), dividing by \( p_{mt}c_{mt} \) and using the two conditions from above:

\[
\frac{P_tC_t}{p_{mt}c_{mt}} = \frac{1}{\omega_m}
\]  

(36)
2. Define an Aggregate Balanced Growth Path
An Aggregate Balanced Growth Path (ABGP) implies:

1. Constant real interest rate: \( r_t = r \ \forall t \).

2. Constant growth of capital per capita: \( \frac{k_{t+1}}{k_t} = 1 + g_k \ \forall t \).

3. Constant growth of GDP per capita: \( \frac{y_{t+1}}{y_t} = 1 + g_y \ \forall t \).

4. Constant capital-to-GDP ratio: \( \frac{K_t}{Y_t} = b \ \forall t \).

5. Constant capital share: \( \frac{r_t K_t}{Y_t} = s \ \forall t \).
3. Show that there is an ABGP
• Assume \( r_t = r \).

• Because \( \frac{k_{xt}}{n_{xt}} = K_t \), with the F.O.C. of the firm at \( t \) and \( t + 1 \) we prove condition 2 of the ABGP:

\[
\left( \frac{K_{t+1}}{K_t} \right)^{\theta-1} \left( \frac{A_{xt+1}}{A_{xt}} \right)^{1-\theta} = 1 \Rightarrow \frac{K_{t+1}}{K_t} = \frac{A_{xt+1}}{A_{xt}} = 1 + \gamma_x
\]  

(37)

• Dividing \( Y_{t+1} \) by \( Y_t \) and using (19) we prove condition 3:

\[
\frac{Y_{t+1}}{Y_t} = \left( \frac{A_{xt+1}}{A_{xt}} \right)^{1-\theta} \left( \frac{K_{xt+1}}{K_{xt}} \right)^{\theta} = (1 + \gamma_x)^{1-\theta} + (1 + \gamma_x)^{\theta} = 1 + \gamma_x
\]  

(38)

• From the previous results, \( Y_t \) and \( K_t \) grow at the same rate \( \Rightarrow \) we prove condition 4:

• Condition 5 follows from the Cobb-Douglas production technology:

\[
\frac{rK_t}{Y_t} = \theta
\]  

(39)
Is $r_t = r \forall t$?

- Dividing the law of motion of capital by $K_t$ we show that $X_t$ also grows at rate $\gamma_x$:

$$\frac{K_{t+1}}{K_t} = (1 - \delta) + \frac{X_t}{K_t} \Rightarrow \frac{X_t}{K_t} = \gamma_x + \delta$$ (40)

- Because both $Y_t$ and $X_t$ grow at rate $\gamma_x$, then $P_tC_t$ grows at the same rate.

- Given this last condition and using the Euler equation (29) we show that $r_t$ is constant:

$$r_{t+1} = r = \frac{1 + \gamma_x}{\beta} - 1 + \delta$$ (41)

- Given a value of $A_{x0}$ and the above condition, a unique value of $K_0$ exists along the ABGP:

$$\bar{K}_0 = \left[ \frac{\beta \theta}{1 + \gamma_x - \beta(1 - \delta)} \right]^{\frac{1}{1-\theta}} A_{x0}$$ (42)
4. Employment and Expenditures Shares along the ABGP
• From the solution to the static household problem:

\[
c_{at} = \frac{\omega_a P_tC_t}{p_{at}} + \bar{c}_a \quad (43)
\]

\[
c_{mt} = \frac{\omega_m P_tC_t}{p_{mt}} \quad (44)
\]

\[
c_{st} = \frac{\omega_s P_tC_t}{p_{st}} - \bar{c}_s \quad (45)
\]

• Because \( \frac{A_{it+1}}{A_{it}} = 1 + \gamma_i, \forall i \in \{a, m, s\} \): \( \frac{p_{it}}{P_t} = \frac{p_{i0}}{P_0} \).

• Expressions (43)–(45) together with constant relative prices and the fact that \( P_tC_t \) grows at rate \( \gamma_x > 0 \) imply:

\[
\frac{c_{at+1}}{c_{at}} < \frac{c_{mt+1}}{c_{mt}} < \frac{c_{st+1}}{c_{st}} \quad (46)
\]
• Thus, it follows that:

\[
\frac{p_{at}c_{at}}{P_tC_t} \downarrow, \quad \frac{p_{mt}c_{mt}}{P_tC_t} =, \quad \frac{p_{st}c_{st}}{P_tC_t} \uparrow
\]  

(47)

• Also, we have that:

\[
\frac{p_{it}c_{it}}{Y_t} = \left(\frac{A_{xt}}{A_{it}}\right)^{1-\theta} k_{it}^{\theta} A_{it}^{1-\theta} n_{it}^{1-\theta} = n_{it}
\]  

(48)

• Hence, because \(Y_t\) grows at rate \(\gamma_x\):

\[
n_{at} \downarrow, \quad n_{mt} =, \quad n_{st} \uparrow
\]  

(49)

• \(X_t\) and \(Y_t\) grow at rate \(\gamma_x\) \(\Rightarrow\) employment and expenditures shares for investment are constant.

• See Proposition 2 of the Handbook Chapter (p. 890) for a condition on \(\bar{c}_s\) that ensures the ABGP is well-defined.